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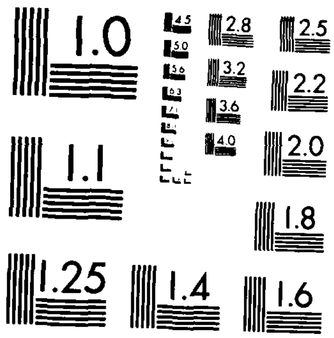

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CHARACTERIZING CONTAINMENT AND RELATED CLASSES OF GRAPHS

by

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Characterizing Containment and Related Classes of Graphs

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ABSTRACT

A graph is a containment graph if one can assign sets to its vertices such that two vertices are adjacent if and only if a set assigned to one contains the set assigned to the other. A containment class of graphs is formed by considering all containment graphs for which the sets assigned to the vertices must be from a prespecified family of sets. We present a characterization of containment classes of graphs as well as characterizations for overlap and disjointedness classes. These results are compared with previous results on intersection classes of graphs.

Characterizing Containment and Related Classes of Graphs

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1. Introduction

This paper characterizes those classes of (finite, simple) graphs which arise as containment graphs of sets of a given type. Our results are strikingly similar to those in [S1] in which intersection classes are characterized.

Let Σ be a collection of nonempty sets. A graph G is called a Σ-*containment graph* provided one can assign sets to the vertices of G so that two vertices are adjacent if and only if a set assigned to one vertex contains the set assigned to the other. More formally: we write $v \sim w$ when vertices v and w are adjacent. A graph G is said to have a Σ-*containment representation* if there exists a one-to-one function $f: V(G) \to \Sigma$ such that $v \sim w$ if and only if $f(v) \subset f(w)$ or $f(v) \supset f(w)$. We denote by $\mathrm{C}(\Sigma)$ the class of all Σ-containment graphs.

It is known [G1] that containment graphs have transitive orientations and every transitively orientable graph is a containment graph. It follows that the family of all transitively orientable graphs is the largest containment class of graphs. Our purpose in this paper is to characterize all containment classes, i.e. if G is a class of graphs, when does there exist a family of sets Σ such that $\mathrm{G} = \mathrm{C}(\Sigma)$.

In section 2 we discuss containment partially ordered sets. In sections 3 and 4 we develop our characterization of containment classes of graphs. In section 5 we drop the one-to-one assumption for containment representations. In section 6 we relate this work to similar work on intersection classes of graphs.

In section 7 we characterize overlap and disjointedness classes of graphs and summarize our results. Finally, in section 8 we consider the problem of characterizing strong containment classes of graphs.

2. Containment Classes of Posets

A *partially ordered set*, or *poset*, is a set, P, together with an irreflexive, antisymmetric, transitive relation $<$. We only consider finite posets in this paper. For a nonempty family of sets, Σ, a poset, P, is said to have a Σ-*containment representation* provided there exists a one-to-one function $f:P \to \Sigma$ such that $x<y$ in P if and only if $f(x) \subset f(y)$. The family of all Σ-containment posets is called a *containment class* of posets and is denoted $\mathbf{C}_P(\Sigma)$. Our first step is toward characterizing $\mathbf{C}(\Sigma)$ classes is to characterize $\mathbf{C}_P(\Sigma)$ classes.

A poset, P, is called an *induced subposet* of P' if $P \subset P'$ and for $x,y \in P$, $x<y$ in P if and only if $x<y$ in P'. Our notation is $P \leq P'$. A class of posets $\mathbf{P}$ is called *monotone* (or *hereditary*) provided $P \in \mathbf{P}$ and $P' \leq P$ imply $P' \in \mathbf{P}$.

Our first necessary condition is:

LEMMA 1. Every containment class of posets is monotone.

PROOF. Obvious. ▪

Our next necessary condition requires the following definition: A *composition sequence* for a class of posets, $\mathbf{P}$, is a sequence of posets $P_1, P_2, P_3, \cdots$ such that (1) each $P_i \in \mathbf{P}$, (2) $P_i \leq P_{i+1}$ for all i, and (3) if $P \in \mathbf{P}$ then $P \leq P_k$ for some k.

LEMMA 2. Every containment class of posets has a composition sequence.

PROOF. Let $\mathbf{P}=\mathbf{C}_P(\Sigma)$ for some collection of sets Σ. Since $\mathbf{P}$ is countable we may assume Σ is countable as well. [One readily checks that for any Σ there exists $\Sigma' \subset \Sigma$ with Σ' countable and $\mathbf{C}_P(\Sigma)=\mathbf{C}_P(\Sigma')$.] Let $\Sigma=\{S_1, S_2, S_3, \cdots\}$. Define $P_k=\{x_1, \cdots, x_k\}$ with $x_i<x_j$ if and only if $S_i \subset S_j$. Clearly each P_k is in $\mathbf{C}_P(\Sigma)$ and

$P_k \leq P_{k+1}$. Further, if $P \in \mathbf{C}_P(\Sigma)$, let $f: P \to \Sigma$ be a containment representation. Let $k = \max\{i: f(x) = S_i,\ x \in P\}$. Clearly $P \leq P_k$. ∎

We now conclude by showing that these two conditions are sufficient.

THEOREM 1. Let **P** be a family of posets. **P** is a containment class if and only if **P** is monotone and has a composition series.

PROOF. The necessity of these two conditions having been proved, we turn to demonstrating their sufficiency.

Let $P_1 \leq P_2 \leq P_3 \leq \cdots$ be a composition sequence for **P**. Since **P** is monotone, we may assume that $|P_k| = k$ and $P_k = \{x_1, \cdots, x_k\}$. Let $S_k = \{y: y \leq x_k \text{ in some } P_j\}$. Let $\Sigma = \{S_1, S_2, S_3, \cdots\}$. One now checks that each P_k is a Σ-containment poset with $f: P \to \Sigma$ by $f(x_i) = S_i$. Since every poset in **P** is an induced subposet of some P_k we have $\mathbf{P} \subset \mathbf{C}_P(\Sigma)$. One verifies the opposite inclusion by considering an arbitrary $P \in \mathbf{C}_P(\Sigma)$. There exists a representation $f: P \to \Sigma$ and put $k = \max\{i: f(x) = S_i,\ x \in P\}$. Clearly $P \leq P_k$ and since **P** is monotone, $P \in \mathbf{P}$. Thus $\mathbf{C}_P(\Sigma) \supset \mathbf{P}$. ∎

3. Preliminary Characterization

In this section we develop our characterization of containment classes of graphs.

Given a poset, P, its *comparability graph*, $\Gamma(P)$, has vertex set P and $x \sim y$ if and only if $x < y$ or $y < x$. It is known that $\Gamma(P)$ is the containment graph of the set of order ideals $\{I(x): x \in P\}$ where $I(x) = \{y: y \leq x\}$. Moreover, if P is a Σ-containment poset with representation $f: P \to \Sigma$ then $\Gamma(P)$ is a Σ-containment graph with the same function as its representation. Finally, the partial order on P induces a natural transitive orientation of $\Gamma(P)$ where $x \to y$ if and only if $x > y$. We can use these known ideas to translate our poset results into graph results by observing $\mathbf{C}(\Sigma) = \Gamma(\mathbf{C}_P(\Sigma))$.

A graph G is an *induced subgraph* of H provided $V(G) \subset V(H)$ and $E(G)=\{xy: x,y \in V(G) \text{ and } xy \in E(H)\}$. We write $G \leq H$ in this case. A class of graphs **G** is *monotone* (or *hereditary*) if $G \leq H \in \mathbf{G}$ implies $G \in \mathbf{G}$.

LEMMA 3. If **G** is a containment class of graphs then **G** is monotone.

PROOF. Obvious. ▪

A *composition sequence* for a class of graphs **G** is a sequence $G_1, G_2, G_3, \cdots$ with (1) each $G_i \in \mathbf{G}$, (2) $G_i \leq G_{i+1}$ for all i, and (3) if $G \in \mathbf{G}$ then $G \leq G_k$ for some k. A composition sequence is called *coherently transitively orientable* provided each graph in the sequence can be transitively oriented so that $x \to y$ in G_i implies $x \to y$ in G_{i+1}.

LEMMA 4. If **G** is a containment class of graphs then **G** has a coherently transitively orientable composition sequence.

PROOF. Let $\mathbf{G} = \mathbf{C}(\Sigma)$ and let $\mathbf{P} = \mathbf{C}_P(\Sigma)$. By lemma 2 **P** has a composition sequence $P_1 \leq P_2 \leq \cdots$. Put $G_i = \Gamma(P_i)$ for all i. It is immediate that $G_1 \leq G_2 \leq \cdots$ is a composition sequence for **G**. Moreover, the natural orientations inherited from the P_i's form a coherent transitive orientation for the sequence. ▪

As in the poset setting, the two conditions together are sufficient.

THEOREM 2. Let **G** be a class of graphs. **G** is a containment class if and only if **G** is monotone and has a coherently transitively orientable composition sequence.

PROOF. The necessity is established in lemmas 3 and 4..

Suppose **G** is monotone and has $G_1 \leq G_2 \leq \cdots$ as its composition sequence. Orient each graph transitively so that the orientations are coherent. Each graph G in **G** can be transitively oriented since $G \leq G_k$ for some k and we orient G according to the orientation of G_k. Thus every graph in **G** has been given a transitive orientation.

A transitively oriented graph G can be considered a poset P with $x > y$ if and only if $x \to y$. [In a sense, $P = \Gamma^{-1}(G)$.] The class **P** of all posets derived in this way

from G is clearly monotone with composition sequence $P_1 \leq P_2 \leq \cdots$. Thus, by theorem 1, P is a containment class of posets, $\mathbf{P}=\mathbf{C}_P(\Sigma)$. It follows easily that $\mathbf{G}=\Gamma(\mathbf{P})$ and so $\mathbf{G}=\mathbf{C}(\Sigma)$. ∎

4. An Alternate Characterization

In the previous section we found necessary and sufficient conditions which a class of graphs must satisfy in order to be a containment class. In some respects this solves the problem at hand. However, the coherence part of the "coherently transitively orientable" condition is, in general, difficult to directly verify. Even if we know that all the graphs in our class are transitively orientable, it is not clear that we can give a coherent orientation to the graphs in the sequence. For example, if we orient G_1, and then attempt to extend that orientation to G_2, and then to G_3, etc. we are doomed to failure. Such a failed construction is depicted in figure 1. One cannot be greedy when constructing a transitive orientation for a graph. In this section we present a more tractable characterization.

A *countable* graph is a graph whose vertex set we allow to be finite or countably infinite. Given a sequence of graphs $G_1 \leq G_2 \leq G_3 \cdots$ we define the limit of this sequence to be the countable graph whose vertex set is the union $\bigcup_{i \geq 1} V(G_i)$ in which $x \sim y$ if and only if they are adjacent in some G_k and one writes $G = \lim_{i \to \infty} G_i$. Note that every finite induced subgraph of the limit graph G is an induced subgraph of some G_k.

It is clear that a transitive orientation for the limit graph implies a coherent transitive orientation for the individual graphs in the sequence. The converse, however, is also true:

LEMMA 5. A sequence of graphs $G_1 \leq G_2 \leq G_3 \leq \cdots$ has a coherent transitive orientation if and only if $\lim_{i \to \infty} G_i$ has a transitive orientation.

PROOF. This follows immediately from the characterization of comparability graphs (which are exactly the transitively orientable graphs) of Gilmore and Hoffman [Gil]. Their characterization, which applies to finite as well as to infinite graphs, states:

A graph is a comparability graph if and only if each odd "cycle" has at least one triangular chord. [Here, a "cycle" may retrace over edges. A "triangular chord" is an edge of the form $v_i v_{i+2}$.]

One need only note that this condition is "locally finite" and a graph fails to be transitively orientable if and only if some finite induced subgraph is not transitively orientable. ▪

We can now state our alternate characterization:

THEOREM 3. A class of graphs is a containment class if and only if it is a monotone class of transitively orientable graphs with a composition sequence.

PROOF. The necessity of these conditions is immediate. Suppose **G** is monotone, contains only transitively orientable graphs and has $G_1 \le G_2 \le \cdots$ as a composition sequence. By theorem 2, we need only show that the sequence is coherently transitively orientable. We know that each G_i has a transitive orientation. Let $G = \lim_{i\to\infty} G_i$. Observe that G must have a transitive orientation, for otherwise one of its finite induced subgraphs would fail to have a transitive orientation. Orient G transitively. Now assign to each G_i the orientation it inherents from G. Clearly the sequence now has a coherent transitive orientation. ▪

5. Noninjective Containment Representations

In the introduction of this paper we required Σ-containment representations $f: V(G) \to \Sigma$ to be one-to-one. In this section we drop this restriction. If $f: V(G) \to \Sigma$ satisfies $v \sim w$ if and only if $f(v) \subset f(w)$ or $f(v) \supset f(w)$ then f is called a *noninjective* Σ-containment representation of G. We denote the class of all

such graphs $\mathbf{C}^*(\Sigma)$. Clearly $\mathbf{C}^*(\Sigma)\supset\mathbf{C}(\Sigma)$. In this section we characterize these noninjective containment classes.

It is clear that every noninjective containment class is monotone. It also follows that it has a composition sequence, although we can no longer rely on posets for our proof. Instead, we may assume that Σ is countable and equals $\{S_1,S_2,S_3,\cdots\}$. We define G_k to be a graph with k^2 vertices v_{ij} with $1\le i,j\le k$. We put $v_{ij}\sim v_{rs}$ if and only if $S_i\subset S_r$ or $S_i\supset S_r$. Clearly each $G_k\in\mathbf{C}^*(\Sigma)$. Furthermore, if $G\in\mathbf{C}^*(\Sigma)$ with $f:V(G)\to\Sigma$ as its noninjective representation, then let $k_1=\max\{i: f(v)=S_i,\ v\in V(G)\}$, $k_2=\max\limits_{i\ge 1}|f^{-1}(S_i)|$ and $k=\max\{k_1,k_2\}$. One now checks that $G\le G_k$, hence $G_1\le G_2\le\cdots$ is a composition sequence.

It now follows by theorem 3 that every noninjective containment class is a containment class of graphs. That is, given Σ, there exists Σ' so that $\mathbf{C}^*(\Sigma)=\mathbf{C}(\Sigma')$. The converse, however, is not true. For example, let $\mathbf{G}$ denote the class of graphs each of whose connected components is a path. By theorem 3, this is a containment class of graphs. It is not, however, an injective containment class for it fails to satisfy one more critical property of noninjective containment classes which now develop.

Given a graph G we can define an equivalence relation on its vertices as follows: If $v,w\in G$ we write $v\approx w$ provided either (1) $v=w$ or else (2) $v\sim w$ and for all other vertices z, $z\sim v$ if and only if $z\sim w$. Next, we define ρG to be the graph G reduced modulo $\approx$, i.e., ρG is an induced subgraph of G formed by taking one vertex in each $\approx$ equivalence class. Since all such induced subgraphs are isomorphic, ρG is uniquely defined. If $H=\rho G$ we also say that G arises from H by *clique vertex expansion* (see [S1]).

LEMMA 6. Let $\mathbf{G}$ be an injective containment class, $\mathbf{G}=\mathbf{C}^*(\Sigma)$. If $G\in\mathbf{G}$ and H arises from G by clique vertex expansion, then $H\in\mathbf{G}$. (We say $\mathbf{G}$ is closed under clique vertex expansion.)

PROOF. We have $G=\rho H$. Let $f:V(G)\to\Sigma$ be a noninjective containment representation. Each vertex v of G corresponds to a $\approx$ equivalence class $\{v_1, \cdots, v_t\}$ in H. Define $f'(v_i)=f(v)$. One now checks that $f':V(H)\to\Sigma$ is a noninjective containment representation. ▪

We now present our characterization.

THEOREM 4. A class of graphs is a noninjective containment class if and only if it is a monotone class of transitively orientable graphs with a composition series which is closed under clique vertex expansion.

PROOF. The necessity of these conditions has just been established. Suppose $\mathbf{G}$ satisfies these conditions. Notice that by theorem 3 $\mathbf{G}$ is a (injective) containment class, $\mathbf{G}=\mathbf{C}(\Sigma)$. We claim that $\mathbf{G}=\mathbf{C}^*(\Sigma)$.

Clearly $\mathbf{G}\subset\mathbf{C}^*(\Sigma)$. Suppose $G\in\mathbf{C}^*(\Sigma)$ and let $f:V(G)\to\Sigma$ be a noninjective containment representation. Since $\rho G\leq G$ it follows that restricting f to $V(\rho G)$ we have a containment representation of ρG. One readily verifies that no pair of distinct vertices of ρG are $\approx$ equivalent. Therefore the restriction of f to $V(\rho G)$ must be one-to-one (for if $f(v)=f(w)$ we would have $v\approx w$). Hence $\rho G\in\mathbf{C}(\Sigma)=\mathbf{G}$. Since $\mathbf{G}$ is closed under clique vertex expansion, it follows that $G\in\mathbf{G}=\mathbf{C}(\Sigma)$, thus $\mathbf{G}\supset\mathbf{C}^*(\Sigma)$. ▪

Although $\mathbf{C}(\Sigma)=\mathbf{C}^*(\Sigma)$ in the proof above, this is not true in general. For example, if $\mathbf{G}$ is the class of graphs each of whose components is a path, we can verify that $\mathbf{G}$ is a containment class but is not a noninjective containment class. In this case $\mathbf{G}=\mathbf{C}(\Sigma)$ for some Σ and $\mathbf{C}(\Sigma)\subset\mathbf{C}^*(\Sigma)$, but $\mathbf{C}(\Sigma)\neq\mathbf{C}^*(\Sigma)$.

6. Containment Classes and Intersection Classes

The following related problem was discussed in [S1] (with slightly different notation). A one-to-one function $f:V(G)\to\Sigma$ is called a Σ-*intersection* representation of G if and only if for all vertices v,w we have $v\sim w$ if and only if

$f(v)\cap f(w)\neq\phi$. G is called a Σ-*intersection* graph and the class of all such graphs is called an *intersection class* and is denoted $\Omega(\Sigma)$. Also, when we relax the one-to-one assumption, we arrive at the notion of noninjective intersection classes, $\Omega^*(\Sigma)$.

THEOREM 5 [#]. A class of graphs is an intersection class if and only if it is monotone and has a composition series. It is a noninjective intersection class if and only if it is an intersection class which is closed under clique vertex expansion. ▪

The striking similarity of this result and those of theorems 3 and 4 gives the following corollary:

COROLLARY. The [noninjective] containment classes of graphs are exactly the [noninjective] intersection classes of graphs whose members are all transitively orientable. ▪

7. Two Further Classes

It is natural when considering containment and intersection representations of graphs to consider two further representation schemes.

First, a graph is said to be a Σ-*overlap* graph, if it has a Σ-*overlap* representation: a one-to-one function $f:V(G)\to\Sigma$ such that $v\sim w$ if and only if $f(v)$ overlaps $f(w)$, i.e. $f(v)\neq f(v)\cap f(w)\neq f(w)$ and $f(v)\cap f(w)\neq\phi$. We denote the family of all Σ-overlap graphs $\mathrm{O}(\Sigma)$. If we allow the function f to not be one-to-one, we arrive at the family of all noninjective Σ-overlap graphs $\mathrm{O}^*(\Sigma)$.

Second, a graph is said to be a Σ-*disjointedness* graph if it has a Σ-*disjointedness* representation: a one-to-one function $f:V(G)\to\Sigma$ such that $v\sim w$ if and only if $f(v)\cap f(w)=\phi$. The family of all Σ-disjointedness graphs is denoted $\Delta(\Sigma)$. By analogy, the family of all noninjective Σ-disjointedness graphs is denoted $\Delta^*(\Sigma)$. We wish to characterize these [noninjective] overlap and dis-

jointedness classes.

It is immediate that a graph G is a Σ-disjointedness graph if and only if $\bar{G}$ is a Σ-intersection graph. Since $G \leq H$ if and only if $\bar{G} \leq \bar{H}$, it follows from theorem 5 that a class of graphs is a disjointedness class if and only if it is monotone and has a composition series. Thus the disjointedness classes are exactly the intersection classes.

In order to characterize noninjective disjointedness classes we need to introduce an alternative notion of vertex expansion. Since assigning the same set to several vertices results in a stable set of vertices with the same neighbors, we define a second notion of vertex equivalence. Given two vertices v,w in a graph G we write $v \equiv w$ if and only if v and w are not adjacent [this includes the case $v = w$] and for all vertices u we have $u \sim v$ if and only if $u \sim w$. Denote by $\rho' G$ the induced subgraph of G formed by taking one vertex per $\equiv$ equivalence class. If $G = \rho' H$ we say H arises from G by ***stable*** vertex expansion. These definitions can be summarized as follows: $v \equiv w$ in H if and only if $v \approx w$ in $\bar{H}$, and $G = \rho' H$ if and only if $\bar{G} = \rho \bar{H}$. The following is now obvious:

THEOREM 6. A class of graphs is a [noninjective] disjointedness class if and only if it is monotone, has a composition sequence [and is closed under stable vertex expansion]. ▪

Next we consider overlap classes of graphs. One can readily check that an overlap class must be monotone and have a composition series. Thus every overlap class must be an intersection class (as well as a disjointedness class). However, every intersection class can be transformed into a overlap class by the following construction. Let $G = \Omega(\Sigma)$ and we may suppose Σ is countable, $\Sigma = \{S_1, S_2, \cdots\}$. Let $X = \{x_1, x_2, \cdots\}$ be a set of elements, in one-to-one correspondence with those in Σ, but no x_i is in any S_j for all i,j, i.e. $X \cap [\bigcup_{i \geq 1} S_i] = \phi$. Let $T_i = S_i \cup \{x_i\}$ and put $\Sigma' = \{T_1, T_2, \cdots\}$. One now verifies that

$G=\Omega(\Sigma)=O(\Sigma')$. Thus the overlap classes are exactly the intersection classes.

Finally we consider noninjective overlap classes. It is immediate that they are monotone, have composition sequences and are closed under stable vertex expansion. Suppose now that a class G has these three properties. It follows that G is an intersection class $\Omega(\Sigma)$. We add an extra element to each of the sets in Σ as before to form Σ'. It is trivial that $G=\Omega(\Sigma)=O(\Sigma')\subset O^*(\Sigma')$. On the other hand, if $G \in O^*(\Sigma')$, then $\rho' G$ is also in that class. In fact $\rho' G \in O(\Sigma')$ since no two vertices of $\rho' G$ can be $\equiv$ equivalent. Thus $\rho' G \in G$ and since G is closed under stable vertex expansion, $G \in G$, hence $G \supset O^*(\Sigma')$. In summary,

THEOREM 7. A class of graphs is a [noninjective] overlap class if and only if it is monotone, has a composition sequence [and is closed under stable vertex expansion]. ▫

There is no difference between overlap classes and disjointedness classes! This is *not* to say $O(\Sigma)=\Delta(\Sigma)$ for arbitrary Σ.

We can summarize our results thus far as follows. The properties we have considered are:

(1) monotone,
(2) has a composition sequence,
(3) contains only transitive orientable graphs,
(4) is closed under clique vertex expansion, and
(5) is closed under stable vertex expansion.

The classes we have discussed and their necessary and sufficient conditions are:

CLASS	PROPERTIES	IF NONINJECTIVE
Containment	1,2,3	1,2,3,4
Intersection	2,3	2,3,4
Overlap	2,3	2,3,5
Disjointedness	2,3	2,3,5

A class of graphs which satisfies all five properties would be simultaneously a containment, intersection, overlap and disjointedness class of both the injective and noninjective varieties. For example, the family of all transitively orientable graphs (or comparability graphs) can be represented in the following ways. Let Σ_1 denote the family of all subtrees of a tree and let Σ_2 denote the family of all cartesian graphs of continuous real value functions (i.e., the curves in the plane which are also referred to as graphs). From [Gav], [G1] and [G2] it follows that:

$$C(\Sigma_1)=C^*(\Sigma_1)=O(\Sigma_1)=O^*(\Sigma_1)=\Omega(\Sigma_1)=\Omega^*(\Sigma_1)=\Delta(\Sigma_2)=\Delta^*(\Sigma_2)$$

all equal the class of transitively orientable graphs!

8. On Characterizing Strong Containment Classes

In [G1] Golumbic defines a notion of *strong* containment class (essentially) as follows: a containment class $C(\Sigma)$ is *strong* if every transitive orientation of each graph in $C(\Sigma)$ is in $P(\Sigma)$. In other words, inherent with every containment representation of a graph is a transitive orientation "induced" by the containment relations of the sets assigned to the vertices. A containment class is strong if all possible transitive orientations for each graph in the class can arise in this manner. Golumbic [G1] shows that the containment graphs of real intervals form a strong containment class, while the containment graphs of edge paths in trees does not form a strong class.

It would be very desirable to have a characterization of strong containment classes similar to our characterization of containment classes. In this section we show that no such characterization is possible! The notion of a containment class of graphs is an intrinsic one: One need only examine the class of graphs G to determine if it is a containment class. However, the notion of strong containment class is an extrinsic one; it depends on the sets in Σ. One cannot say whether or not a containment class G is strong or not. We prove this by showing

that there exist two families of sets Σ and Σ' with $\mathbf{C}(\Sigma)=\mathbf{C}(\Sigma')$ where $\mathbf{C}(\Sigma)$ is not a strong containment class but $\mathbf{C}(\Sigma')$ is a strong containment class.

Let Σ be the family of all edge paths in trees (see [#]). Let $\mathbf{G}=\mathbf{C}(\Sigma)$. We know that $\mathbf{C}(\Sigma)$ is not a strong containment class. We can easily check that $\mathbf{G}$ is closed under disjoint union, i.e. if G and H are in $\mathbf{G}$ then their disjoint union is also in $\mathbf{G}$. Let $\mathbf{P}$ denote the class of posets formed by taking all possible transitive orientations of all graphs in $\mathbf{G}$. Clearly $\mathbf{P}$ is monotone, and closed under disjoint union of posets. Let P_i denote the disjoint union of all posets in $\mathbf{P}$ with at most i elements. It follows that $P_1 \leq P_2 \leq \cdots$ is a composition sequence for $\mathbf{P}$, hence there exists a set Σ' with $\mathbf{P}=\mathbf{C}_P(\Sigma')$. One now checks that $\mathbf{G}=\mathbf{C}(\Sigma)=\mathbf{C}(\Sigma')$ and that $\mathbf{C}(\Sigma')$ is a strong containment class!

Thus it is impossible to characterize classes of graphs $\mathbf{G}$ as strong or not strong containment classes; the notion is not intrinsic to the class $\mathbf{G}$, but is extrinsic, i.e., it depends on how the class is represented. This leaves us with a more difficult problem: Characterize those family of sets Σ such that $\mathbf{C}(\Sigma)$ is (or is not) a strong containment class.

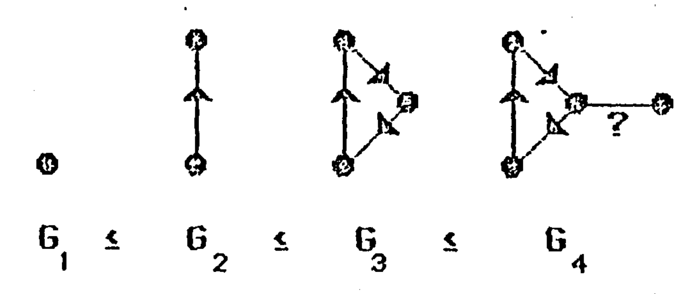

Figure 1.

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20. ABSTRACT

This report presents a characterization of containment graphs for overlap and disjointedness classes. These results are compared with previous results on intersection classes of graphs.

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